International Journal of Theoretical Physics, Vol. 43, No. 9, September 2004 (© 2004)

Hamiltonian Formalism of the Landau–Lifschitz Equation for a Spin Chain with an Easy Plane

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With a suitable gauge transformation, the Hamiltonian formalism of the Landau–Lifschitz equation for a spin chain with an easy plane is established by standard procedure. Action-angle variables are obtained and the canonical equation is given.

KEY WORDS: solitons; inverse problems; spin chain models.

1. INTRODUCTION

The Landau–Lifschitz (L–L) equation (Landau and Lifschitz, 1935) for a spin chain is a typical complete integrable equation in $1 + 1$ dimension with its physical application. The L–L equation for an isotropic spin chain, the simplest case, was solved by an inverse scattering transform (IST) with a little modification (Takhtajan, 1977; Laksmanan, 1977). As known, in the inverse transform equation, a redundant factor k^{-1} needs to be introduced to ensure vanishing contribution of integral along the big arc in k-plane as the spectral parameter $|k| \to \infty$. A series of conserved quantities is obtained through expanding $a(k)$ by k^{-1} (Fogedby, 1980). The zeroth-order term in this series does not vanish unlike that in the well-known non-linear Schrödinger equation (NLSE). In addition, the Hamiltonian can not be derived by the first-order term. This difficulty was overcome in consideration of the gauge equivalence between the NLSE and the L–L equation for an isotropic spin chain (Zakharov and Takhtajan, 1979; Faddeev and Takhtajan, 1987). By using the reverse one of the gauge transformation, one can obtain the conservation laws of the L–L equation for an isotropic spin chain from those of the NLSE.

For the L–L equation for a spin chain with axial symmetry (Tjio and Wright. 1977), in order to derive the right conservation laws and obtain a valuable guide to

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0020-7748/04/0900-1931/0

construct inverse scattering transform, one tried to look for the gauge equivalent equation but failed. In 1995, the L–L equation for a spin chain with an easy plane was solved by Darboux matrix method and avoid problem (Nian-Ning Huang *et al.*, 1995). It means that it is necessary to analyze the essence of the gauge equivalence of the L–L equation for the isotropic spin chain with the NLSE. We find the essence is: the spinning vector (in the leading power of k in the first one of Lax pair of the L–L equation for an isotropic spin chain) is turned to one of three-axis in spin space by a gauge transformation. Since the spinning vector of L–L equation for a spin chain with an easy plane is the same as that of L–L equation for an isotropic spin chain, the same procedure can be applied to the spin chain with an easy plane.

In this paper, we give the explicit expression of the gauge transformation, derive the right conservation laws and construct the Hamiltonian formalism of the L–L equation for a spin chain with an easy plane. The coordinate and spectral parameter representations of the Hamiltonian are deduced. The spectral parameter expressions of a series of conserved quantities is then derived and, by introducing a suitable gauge transformation, we find the coordinate expressions, one of them is just the required Hamiltonian. And finally, the discrete spectrum part of the Hamiltonian formalism is constructed.

2. LIE–POISSON BRACKET AND THE COORDINATE REPRESENTATION OF THE HAMILOTONIAN

The L–L equation for a spin chain with an easy plane is

$$\vec{S}_t = \vec{S} \times \vec{S}_{xx} + \vec{S} \times J\vec{S} \tag{1}$$

where $|\vec{S}| = 1$, and $J = \mathrm{diag}(J_1, J_2, J_3)$ characterizing the magnetic properties of the spin chain. For the spin chain, $\vec{S}$ satisfies Lie–Poisson bracket

$$\{S_a(x), S_b(y)\} = -\epsilon_{abc} S_c(x)\delta(x - y) \tag{2}$$

where ε_{abc} is totally skew-symmetric rank 3 tensor, $a, b, c = 1, 2, 3$ and having two indices means we are summing over this index. Then the Lie–Poisson bracket of the Q and R as the function of $S_a(x)$ can be written.

In terms of Lie–Poisson bracket, the Hamiltonian equation of L–L equation is

$$\vec{S}_{at} = \{\vec{S}_a, H\} \tag{3}$$

where H is determined uniquely by

$$H = \int dx\mathcal{H}(x), \mathcal{H}(x) = \frac{1}{2}S_{bx}S_{bx} - \frac{1}{2}J_b S_b S_b \tag{4}$$

For a spin chain with an easy plane, $J_2 < J_1 = J_3$, one can set $J_2 = 0$ and obtain $J_1 = J_3 = J > 0$.

3. THE LIE–POISSON BRACKETS BETWEEN ELEMENTS OF MONODROMY MATRIX

For a spin chain with an easy plane, the pair of compatibility conditions can be written as

$$L = -i \sum_a u_a(\zeta) S_a \sigma_a, \tag{5}$$

and

$$M = -i \sum_{a,b,c} u_a(\zeta) S_b S_{cx} \sigma_a \epsilon_{abc} + i \sum_{a,b,c} u_b(\zeta) u_c(\zeta) S_a \sigma_a |\epsilon_{abc}|, \tag{6}$$

where $u_1 = u_3 = \kappa$, $u_2 = \lambda$ and $\lambda^2 - \kappa^2 = \rho^2$, here ρ^2 is a constant. Introduce affine parameter ζ so that $\lambda = \frac{1}{2}(\zeta + \rho^2 \zeta^{-1})$, $\kappa = \frac{1}{2}(\zeta - \rho^2 \zeta^{-1})$ are single-valued functions of ζ.

One can assume that $\vec{S} \to (0, 0, 1)$ as $|x| \to \infty$, hence the first compatibility equation

$$\partial_x F(x, \zeta) = L(x, \zeta) F(x, \zeta) \tag{7}$$

has an asymptotic solution, the so-called free Jost solution, $E(x, \zeta) = e^{-i\kappa x \sigma_3}$.

Then the Jost solutions are defined by

$$\begin{aligned} \Psi(x, \zeta) &= (\tilde{\psi}(x, \zeta), \psi(x, \zeta)) \to e^{-i\kappa x \sigma_3}, \quad \text{as } x \to \infty \\ \Phi(x, \zeta) &= (\phi(x, \zeta), \tilde{\phi}(x, \zeta)) \to e^{-i\kappa x \sigma_3}, \quad \text{as } x \to -\infty \end{aligned} \tag{8}$$

Introducing a monodromy matrix $T(\zeta)$ by

$$\Phi(x, \zeta) = \Psi(x, \zeta) T(\zeta), \qquad T(\zeta) = \begin{pmatrix} a(\zeta) & -\tilde{b}(\zeta) \\ b(\zeta) & \tilde{a}(\zeta) \end{pmatrix} \tag{9}$$

we can have

$$a(\zeta) = \psi(x, \zeta)^T (-i\sigma_2) \phi(x, \zeta) \tag{10}$$

and

$$b(\zeta') = \tilde{\psi}(x, \zeta')^T (i\sigma_2) \phi(x, \zeta'). \tag{11}$$

Following a variation procedure, we obtain

$$\frac{\delta a(\zeta)}{\delta S_a(z)} = -u_a \psi(z, \zeta)^T \sigma_2 \sigma_a \phi(z, \zeta) \tag{12}$$

and

$$\frac{\delta b(\zeta')}{\delta S_b(z)} = u'_a \tilde{\psi}(z, \zeta')^T \sigma_2 \sigma_b \phi(z, \zeta). \tag{13}$$

Hence, (12) and (13) lead to

$$\{a(\zeta), b(\zeta')\} = \epsilon_{abc} \int dz\, u_a u_b' \psi(z,\zeta)^T \sigma_2 \sigma_a \phi(z,\zeta') \tilde{\psi}(z,\zeta')^T \sigma_2 \sigma_b \phi(z,\zeta) S_c(z) \tag{14}$$

This integral will be obtained if the integrand is a complete differential. Notice the Jost solutions, after some manipulation we find that the integrand of the right hand of (14) is

$$\begin{aligned}\{a(\zeta), b(\zeta')\} &= \frac{1}{2}\frac{\kappa\lambda' + \kappa'\lambda}{\kappa - \kappa'}(\psi(z,\zeta)^T \sigma_2 \tilde{\psi}(z,\zeta')\phi(z,\zeta')^T \sigma_2 \phi(z,\zeta))\Big|_{z=-L}^{z=L} \\ &\quad + \frac{1}{2}\frac{\kappa\lambda' - \kappa'\lambda}{\kappa + \kappa'}(\psi(z,\zeta)^T \tilde{\psi}(z,\zeta')\phi(z,\zeta')^T \phi(z,\zeta))\Big|_{z=-L}^{z=L}. \end{aligned} \tag{15}$$

From (8), one proves that

$$\begin{aligned}\{a(\zeta), b(\zeta')\} &= -\frac{\kappa\lambda' + \kappa'\lambda}{1+\rho^2\zeta^{-1}\zeta'^{-1}}\frac{1}{\zeta - \zeta' - i0}a(\zeta)b(\zeta') \\ &\quad + \frac{\kappa\lambda' - \kappa'\lambda}{1+\zeta^{-1}\zeta'}\frac{1}{\zeta - \rho^2\zeta'^{-1} + i0}a(\zeta)b(\zeta') \end{aligned} \tag{16}$$

Similarly we obtain

$$\begin{aligned}\{\tilde{a}(\zeta), b(\zeta')\} &= -\frac{\kappa\lambda' + \kappa'\lambda}{1+\rho^2\zeta^{-1}\zeta'^{-1}}\frac{1}{\zeta - \zeta' - i0}\tilde{a}(\zeta)b(\zeta') \\ &\quad + \frac{\kappa\lambda' - \kappa'\lambda}{1+\zeta^{-1}\zeta'}\frac{1}{\zeta - \rho^2\zeta'^{-1} + i0}\tilde{a}(\zeta)b(\zeta') \end{aligned} \tag{17}$$

Consequently, (16) and (17) yield

$$\begin{aligned}\{|a(\zeta)|^2, b(\zeta')\} &= \kappa'\zeta' i2\pi\delta(\zeta - \zeta')|a(\zeta)|^2 b(\zeta') \\ &\quad - \kappa'\rho^2\zeta'^{-1} i2\pi\delta(\zeta - \rho^2\zeta'^{-1})|a(\zeta)|^2(\zeta') \end{aligned} \tag{18}$$

4. THE CONTINUOUS PART OF SPECTRAL REPRESENTATION OF HAMILTONIAN

The Hamiltonian of the continuous spectrum will be discussed. From the inverse scattering method, we see that $a(\zeta), \tilde{a}(\zeta)$ are independent of the time variable t and that $b(\zeta), \tilde{b}(\zeta)$ depend on the time variables t in the following manner. When $|x| \to \infty$, $M \to M_0 = i2\kappa\lambda\sigma_3$, we have

$$b(t,\zeta) = b(0,\zeta)e^{-i4\kappa\lambda t}, \qquad a(t,\zeta) = a(0,\zeta) \tag{19}$$

Therefore, the action variables $P(\zeta)$ must be a function of $a(\zeta)$ and $\tilde{a}(\zeta)$. Assume that the action variable is

$$P(\zeta) = F(|a(\zeta)|^2) \tag{20}$$

where F is a function to be determined, the angle variable $Q(\zeta)$ is

$$Q(\zeta) = \arg b(\zeta) = \frac{1}{i}\ln b(\zeta), \qquad Q(\zeta, t) = Q(\zeta, 0) - 4\kappa\lambda t \tag{21}$$

Their Lie–Poisson bracket must be

$$\{P(\zeta), Q(\zeta')\} = -\delta(\zeta - \zeta') \tag{22}$$

The form of the function F can be determined from (22).

For the spin chain with an easy plane, (5) leads to $L(\zeta) = \sigma_2 L(\rho^2\zeta^{-1})\sigma_2$ and we find

$$a(\zeta) = \tilde{a}(\rho^2\zeta^{-1}) \tag{23}$$

Thus only the case of $|\zeta| > p$ needs to be treated. Now only the first term of (18) is needed, so that

$$\{F(|a(\zeta)|^2), \ln b(\zeta')\} = F'(|a(\zeta)|^2)|a(\zeta)|^2 \pi 2\kappa\zeta\delta(\zeta - \zeta') \tag{24}$$

With the aid of (21) and (22), integrating (24) yields the form of the function F as follows

$$P(\zeta) = F(|a(\zeta)|^2) = -\frac{1}{\pi 2\kappa\zeta}\ln|a(\zeta)|^2 \tag{25}$$

So, by using (23), the continuous spectrum of the Hamiltonian can be uniquely expressed as

$$\begin{aligned} H &= -\int_{|\zeta|>p} d\zeta 4\kappa\lambda P(\zeta) = \frac{1}{\pi}\int_{|\zeta|>p} d\zeta(1 + \rho^2\zeta^{-2})\ln|a(\zeta)|^2 \\ &= \frac{1}{\pi}\int_{-\infty}^{+\infty} d\zeta \ln|a(\zeta)|^2 \end{aligned} \tag{26}$$

5. ASYMPTOTIC BEHAVIOR OF THE JOST SOLUTIONS

For the two components of Jost solution, the first Lax equation (7) can be expressed as

$$v_{1x} = L_{11}v_1 - L_{12}v_2, \qquad v_{2x} = L_{21}v_1 + L_{22}v_2 \tag{27}$$

where v is $\psi(x, \zeta)$ and L in (5). Eliminating v_1, we have

$$(v_{2x} - L_{22}v_2)_x - \left(\frac{1}{(L_{21})_x} + L_{11}\right)(v_{2x} - L_{22}v_2) - L_{12}L_{21}v_2 = 0 \tag{28}$$

Set $v_2 = e^{i\frac{1}{2}\zeta x+g}$ and introduce

$$g_x \equiv \mu = \mu_0 + \mu_1(i\zeta)^{-1} + \cdots \tag{29}$$

as $|\zeta| \to \infty$. Considering the asymptotic expansion, v_2 and substituting for L, g_x by (5), (29), respectively in (28), $\mu_0 \neq 0$ is derived. Hence we conclude that, as in the isotropic case, it is impossible to obtain right conserved quantities in the forms of integral with respect to x.

6. A GAUGE TRANSFORMATION

Since a gauge transformation has no effect on the monodromy matrix, we choose a gauge B which turns the spin in the leading power of spectral parameter in the first one of Lax pair into one of three-axis in spin space, namely

$$BS_a\sigma_a B^{-1} = \sigma_3. \tag{30}$$

Under B, L is transformed to

$$L' = BLB^{-1} + B_x B^{-1} \tag{31}$$

Taking polar coordinates $(1, \theta, \varphi)$ in spin space we choose

$$B = e^{i\frac{1}{2}\sigma_2 f}A,\ A(x,t) = e^{i\frac{1}{2}\sigma_2\theta}e^{i\frac{1}{2}\sigma_3\varphi}. \tag{32}$$

Thus

$$ALA^{-1} = -i\frac{1}{2}\zeta\sigma_3 - i\frac{1}{2}\rho^2\zeta^{-1}\big(-S_1^2 + S_2^2 - S_3^2\big)\sigma_3 + \zeta^{-1}W_1 \tag{33}$$

where

$$W_1 = -\rho^2 \sin\theta \sin\varphi(i\sigma_2\cos\varphi + i\sigma_1\sin\varphi\cos\theta) \tag{34}$$

the element has zeros on the diagonal. But $A_x A^{-1}$ is

$$A_x A^{-1} = i\frac{1}{2}\sigma_2\theta_x + i\frac{1}{2}\sigma_3\varphi_x e^{-i\sigma_2\theta} \tag{35}$$

which has non-vanishing diagonal elements. So we choose another gauge transformation B which rotates AS around the three-axis by the angle of f. Thus we get

$$B_x B^{-1} = i\frac{1}{2}\sigma_3 f_x + e^{i\frac{1}{2}\sigma_3 f}A_x A^{-1}e^{-i\frac{1}{2}\sigma_3 f} \tag{36}$$

and the diagonal elements of $B_x B^{-1}$ vanish as long as $i\frac{1}{2}f_x + i\frac{1}{2}\varphi_x\cos\theta\sigma_3 = 0$. This means that f can be determined if φ and $\cos\theta$ are given explicitly. Now L is transformed into

$$L' = -i\frac{1}{2}\zeta\sigma_3 - i\frac{1}{2}\rho^2\zeta^{-1}\big(-S_1^2 + S_2^2 - S_3^2\big)\sigma_3 + U \tag{37}$$

where

$$U = B_x B^{-1} + \zeta^{-1} W_1 \tag{38}$$

the element has zeros on the diagonal.

7. CONSERVED QUANTITIES

Replacing L by L', from (28) one obtains

$$\mu_0 = 0, \qquad \mu_1 = -|u|^2 - \frac{1}{2}w, \ldots. \tag{39}$$

In other words

$$\mu_1 = -|u|^2 - \frac{1}{2}\rho^2\big(-S_1^2 + S_2^2 - S_3^2\big) = -|u|^2 + \frac{1}{4}J_1 S_1^2 + \frac{1}{4}J_3 S_3^2 \tag{40}$$

Here constant terms are neglected.

According to Tacktajan [?], $4|u|^2 = S_{ax}S_{ax}$. Therefore, (40) becomes

$$\mu_1 = -\frac{1}{4}\big(S_{ax}S_{ax} - J\big(S_1^2 + S_3^2\big)\big) \tag{41}$$

Because of

$$\ln a(\zeta) = -\int_{-\infty}^{\infty} dx\{\mu_0 + \mu_1 (i\zeta)^{-1} + \cdots\}, \tag{42}$$

we get

$$I_0 = 0, \qquad I_1 = \int_{-\infty}^{\infty} dx \left(\frac{1}{4}S_{ax}S_{ax} - \frac{1}{4}JS_1^2 - \frac{1}{4}JS_3^2\right), \tag{43}$$

Now $I_1 = \frac{1}{2}H$ follows from (4) and (43).

8. DISPERSION RELATION

In the inverse scattering transform $a(\zeta)$ is independent of t, and the dispersion relation is given by

$$\ln a(\zeta) = \frac{1}{2i\pi}\int_{-\infty}^{\infty} d\zeta' \frac{\ln |a(\zeta')|^2}{\zeta' - \zeta} \tag{44}$$

so that

$$\ln a(\zeta) = \sum_{j=0}^{\infty} I_j (i\zeta)^{-j} \quad \text{as } |\zeta| \to \infty \tag{45}$$

where I_j are conserved quantities. From (26) and (45) one can see that $H = 2I_1$ and $I_0 = 0$.

9. THE DISCRETE SPECTRUM

For simplicity we assume that $a(\zeta)$ has only a pair of zeros. A standard procedure yields

$$a_d(\zeta) = \frac{\zeta - \zeta_1}{\zeta - \bar{\zeta}_1}\frac{\zeta - \zeta_{1'}}{\zeta - \bar{\zeta}_{1'}}, \quad \zeta_{1'} = \rho^2 \bar{\zeta}_1^{-1} \tag{46}$$

where $\zeta_{1'} = \rho^2 \bar{\zeta}_1^{-1}$ from (23). When $|\zeta| \to \infty$, $a_d(\zeta) \to 1$. So we have

$$\ln a_d(\zeta) + (\bar{\zeta}_1 - \zeta_1)\zeta^{-1} + \cdots + (\bar{\zeta}_{1'} - \zeta_{1'})\zeta^{-1} + \cdots \tag{47}$$

The discrete part of Hamiltonian H_c is then

$$H_d = 2I_{\mathrm{d}1} = -i2(\zeta_1 + \zeta_{1'} - \bar{\zeta}_1 - \bar{\zeta}_{1'}) = -i4(\kappa_1 - \bar{\kappa}_1) \tag{48}$$

10. ACTION-ANGLE VARIABLES IN DISCRETE SPECTRUM

According to the inverse scattering transform ζ_n are independent of t and $b_n(t)$ has a phase of the form $-i4\lambda_n\kappa_n t$. We assume that the action variables P_n

$$P_n = G(\zeta_n) \tag{49}$$

where G is a function to be determined. Let the angle variables Q_n be

$$Q_n = \ln b_n = \ln |b_n| + i \arg b_n \tag{50}$$

The Lie–Poisson brackets containing b_n can be derived as follows. Suppose $a(\zeta)$ has a pair of zeros ζ_1, $\zeta_{1'}$, as in (46), and by setting $\zeta' = \zeta_1$ in (16), we have

$$\{a(\zeta), b_1\} = -\frac{\kappa\lambda_1 + \kappa_1\lambda}{1 + \rho^2\zeta^{-1}\zeta_1^{-1}}\frac{1}{\zeta - \zeta_1}b_1 + \frac{\kappa\lambda_{1'} - \kappa_{1'}\lambda}{1 + \zeta^{-1}\zeta_1}\frac{1}{\zeta - \bar{\zeta}_{1'}}b_1 \tag{51}$$

From (46) we obtain

$$\{\ln a(\zeta), b_1\} = -\frac{\{\zeta_1, b_1\}}{\zeta - \zeta_1} - \frac{\{\zeta_{1'}, b_1\}}{\zeta - \zeta_{1'}} + \frac{\{\bar{\zeta}_1, b_1\}}{\zeta - \bar{\zeta}_1} + \frac{\{\bar{\zeta}_{1'}, b_1\}}{\zeta - \bar{\zeta}_{1'}} \tag{52}$$

Comparing (51) and (52) we find

$$\{\zeta_1, b_1\} = \kappa_1\zeta_1 b_1, \qquad \{\bar{\zeta}_{1'}, b_1\} = \kappa_1\bar{\zeta}_{1'}b_1, \tag{53}$$

Therefore

$$\{H_d, b_1\} = 2\{\zeta_1 - \bar{\zeta}_{1'}, b_1\} = 4\kappa_1\lambda_1 b_1 \tag{54}$$

From (49) and (50) we arrive at

$$P_1 = G(\zeta_1), \qquad Q_1 = \ln b_1 \tag{55}$$

$$\{P_1, Q_1\} = \frac{G'}{b_1}\{\zeta_1, b_1\} = \frac{G'}{b_1}\kappa_1\zeta_1 b_1 = -1 \tag{56}$$

$$\frac{d}{d\zeta_1}G(\zeta_1) = -\kappa_1^{-1}\zeta_1^{-1}$$
$$G(\zeta_1) = -2\int d\zeta_1\big(\zeta_1^2 - \rho^2\big)^{-1} = \frac{2}{\rho}\text{Arth}\frac{\zeta_1}{\rho} \tag{57}$$

Similarly we obtain

$$G(\zeta_{1'}) = -2\int d\zeta_{1'}\big(\zeta_{1'}^2 - \rho^2\big)^{-1} = \frac{2}{\rho}\text{Arth}\frac{\zeta_{1'}}{\rho}$$

which when combined with (57) implies

$$H_d = -i2(1-\rho)th\frac{\rho}{2}(G_1 - G_{1'}) + c.c. = -i2(\zeta_1 - \bar{\zeta}_{1'} + c.c. \tag{58}$$

This is exactly equation (48). If $a(\zeta)$ has many pairs of zeros the above expression will involve more terms.

11. CONCLUDING REMARK

As $\rho \to 0$, that is, when the anisotropy vanishes, it is obvious that (5) reduces to the expression of $L = -i\sum_a kS_a\sigma_a$ for an isotropic spin chain (where k is the spectral parameter). Similarly one can see that (16)–(18), (25) and (57) reduce to the corresponding expressions for the isotropic spin chain.

In this paper, with the formal procedure we show the Hamiltonian formalism of the Landau–Lifschitz equation for a spin chain with an easy plane. In deriving the coordinate expression of conserved quantities, we first apply a gauge transformation to overcome technical difficulties.

ACKNOWLEDGMENT

The project is supported by the National Natural Science Foundation of China, No. 10375041.

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